$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial \theta}+\frac{\partial}{\partial \alpha}\right)P^2(t,\theta,\alpha)=\Phi(\theta)W\Phi^*(\alpha)$$
$$-P^1(t,\theta)C^*V^{-1}CP^{1*}(t,\alpha)$$
$$P^2(0,\theta,\alpha)=P^2(t,\theta,-\epsilon)=P^2(t,-\epsilon,\alpha)=0$$
$$-\epsilon\le\theta\le 0,\qquad -\epsilon\le\alpha\le 0,\qquad 0<t\le T \qquad (19)$$

Equations (14–19) determine the Kalman filtering algorithm for best estimates $\hat{x}^*(t)$ and $\hat{\xi}(t)$ in the wide-band noise driven filtering problem. The system Riccati equation (17–19) are independent of observations and so the solution can be computed beforehand. Therefore, Eqs. (14–19) determine two linear transformations which transform the observations into best estimates $\hat{x}^*$ and $\hat{\xi}$.

The separation principle and Kalman filtering results described earlier lead us to a synthesis of optimal control in the linear stochastic regulator problem for wide-band noise case. In fact it is given by the formulas (9), (11), (13), and (14–19).

Computational Aspects

To apply the previous results, one must model a wide-band noise in the form of Eq. (4). In practice a wide-band noise is characterized by its autocovariance function. Therefore, the problem concerns modeling a wide-band noise φ in the form of Eq. (4), the autocovariance function Λ being given. If we suppose that the underlying white noise w in Eq. (4) has the covariance $\text{cov}[w(t), w(s)] = I\delta(t-s)$ where I is an identity matrix then, from Eq. (3), we obtain that Φ is a solution of the equation

$$\int_{s-\epsilon}^{0}\Phi(\theta-s)\Phi^*(\theta)\,\mathrm{d}\theta=\Lambda(s),\qquad 0\le s\le\epsilon \qquad (20)$$

Equation (20) is a convolution equation. It can be solved by direct and inverse Fourier or Laplace transforms, readily performed on computers using efficient algorithms.

In the second stage the system Riccati equations (17–19) can be solved. They contain an ordinary differential equation [see Eq. (17)] as well as partial differential equations [see Eqs. (18) and (19)]. The next stage being calculations of the best estimates $\hat{x}^*$ and $\hat{\xi}$ also contains the ordinary differential equations [see Eqs. (14) and (15)] as well as a partial differential equation [see Eq. (16)]. Note that the availability of partial differential equations in the Kalman filtering algorithm for the wide-band noise driven linear filtering problem is a result of a special form of wide-band noise used in the form of distributed delay of white noise. Such partial differential equations are also obtained in Kalman filtering algorithms for white noise driven linear filtering problems with the signal process described by linear differential delay equations.

Finally, if one applies the linear regulator result to them one must also solve the Riccati equation (11) and use Eq. (9) which does not contain any part with a partial differential equation. Note that the solution of Eq. (11) takes place in Eq. (15). Therefore, Eq. (11) must be solved before the calculation of $\hat{\xi}$.

Generalizations

The results on wide-band noise presented here are a simple case of the general theory considered in Bashirov.[5,6] First, they are proved for the nonstationary case and for infinite dimensional systems and, therefore, contain the case of an ordinary differential as well as the cases of partial differential equations of the parabolic and hyperbolic types, differential delay equations, etc. The separation principle holds for the general case of arbitrary dependent noises. The filtering result has a modification to the case when the state, as well as the observations, are subjected to the action of the sum of correlated white and wide-band noises. Moreover, if we consider the point delay of white noise rather than distributed one, then we obtain the control and filtering problems with dependent white noise. The Kalman filtering theory for such linear filtering problems is developed in Bashirov and Mishne.[9]

Conclusions

The separation principle and Kalman filtering for the white noise driven linear systems were the underlying results of construction of complex design. This Note extends these results for the wide-band noise case.

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Reduced Order Proportional Integral Observer with Application

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I. Introduction

A MAJORITY of the controllers proposed for the multivariable robust servomechanism problem employ integral action along with a state feedback approach.[1–6] Since the states in multivariable systems are not always available for feedback purposes, the estimation of the missing state variables is very much of interest. Although the robust servomechanism has received considerable attention in the past, the estimation problem has only been mentioned by a few authors.[4,7]

In this Note a systematic approach for designing a proportional integral observer (PIO) is proposed that estimates both the state and the constant disturbance affecting the plant. This problem is a special case of estimating the state of a dynamical system driven by unknown time varying disturbances using an unknown input observer (UIO) theory. The UIO theory has been the subject of a number of research studies in the past (see Refs. 8 and 9 and references cited therein). The UIO approach is more general since it can handle time varying disturbances.

Received June 11, 1992; revision received Oct. 6, 1992; accepted for publication Nov. 18, 1992.

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However, there it is assumed that the disturbances can only effect the plant, and furthermore they cannot be estimated.

The idea of using an observer with integral action has been implicitly suggested in Refs. 4 and 7. Shafai and Carroll[10] and Beale and Shafai[11] were the first to formally use the PIO for robustness recovery in control systems employing an estimator in the loop. It was shown there that the addition of the integral action in the observer would provide an additional degree of freedom for recovering the traditional stability margins of the linear quadratic regulators.

This Note provides additional insight to the design of PIOs; and, unlike Refs. 10 and 11, this Note deals with the problem of state as well as disturbance estimation in multivariable systems. In addition to servomechanism problems, this approach may also prove useful in instrument fault detection and identification.[12]

II. Proportional Integral Observer Design

Consider a linear time invariant dynamical system described as

$$\dot{x} = Ax + Bu + Dv \tag{1}$$

$$y = Cx + Ev \tag{2}$$

where

$$C = [0 \quad I]$$

where $x \in \Re^n$ is the state, $u \in \Re^q$ the control, $v \in \Re^m$ the unknown constant disturbance input, and $y \in \Re^p$ the output of the system. It is assumed that the measurement distribution matrix C is in the form just given or that the system has been transformed via a similarity transformation (see Ref. 13) so that the matrix C is in this desired form.

The estimation problem is to design a reduced order estimator capable of estimating the $(n-p)$ unavailable state variables as well as the disturbance vector v. The following theorem indicates how to design such an estimator.

Theorem 1: The state of the dynamical system in Eq. (1), along with the constant unknown disturbance acting on the plant, and the measurement can be estimated using a reduced order estimator of the form

$$\dot{w} = Fw + Gu + Hy \tag{3}$$

$$F = \begin{bmatrix} A_{11} - M_1 A_{21} & (D_1 - M_1 D_2) + (M_1 A_{22} - A_{12})E \\ -M_2 A_{21} & M_2(A_{22}E - D_2) \end{bmatrix} \tag{4}$$

$$G = \begin{bmatrix} B_1 - M_1 B_2 \\ -M_2 B_2 \end{bmatrix} \tag{5}$$

$$H = L + FM \tag{6}$$

$$L = \begin{bmatrix} A_{12} - M_1 A_{22} \\ -M_2 A_{22} \end{bmatrix} \tag{7}$$

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix} \tag{8}$$

if and only if the pair

$$\left\{ \begin{bmatrix} A_{21} & D_2 - A_{22}E \end{bmatrix}, \begin{bmatrix} A_{11} & D_1 - A_{12}E \\ 0 & 0 \end{bmatrix} \right\} \tag{9}$$

is completely observable. Furthermore, in such a case, the estimator's gain M can be appropriately selected so that its poles can be placed arbitrarily (within the usual complex conjugacy condition).

Proof: Consider decomposing the system in Eqs. (1) and (2) in the following way:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} v \tag{10}$$

$$y = x_2 + Ev \tag{11}$$

and define vectors q and r as

$$q = \dot{x}_2 - A_{22}x_2 - B_2 u = A_{21}x_1 + D_2 v$$
$$= \dot{y} - A_{22}y + A_{22}Ev - B_2 u \tag{12}$$

and

$$r = A_{12}x_2 + B_1 u = A_{12}y - A_{12}Ev + B_1 u \tag{13}$$

Note that in expressions (12) and (13) all other signals but v are known. Thus, using an estimate of v we get

$$q \cong \dot{y} - A_{22}y + A_{22}E\hat{v} - B_2 u \tag{14}$$

$$r \cong A_{12}y - A_{12}E\hat{v} + B_1 u \tag{15}$$

Using this in Eq. (10), we get

$$\dot{x}_1 = A_{11}x_1 + D_1 v + r$$

$$q = A_{21}x_1 + D_2 v$$

The following estimator's dynamics which involves an integral and a proportional term involving the difference between q and its estimate is proposed:

$$\dot{\hat{x}}_1 = A_{11}\hat{x}_1 + D_1 M_2 \int (q - A_{21}\hat{x}_1 - D_2\hat{v})\, dt$$
$$+ M_1(q - A_{21}\hat{x}_1 - D_2\hat{v}) + r \tag{16}$$

where M_i are the estimator's gains. Upon substitution for q, r, using Eqs. (14) and (15) the following is obtained:

$$\dot{\lambda} = F\lambda + Gu + M\dot{y} + Ly \tag{17}$$

where F, G, and L are defined in Eqs. (4), (5), and (7), respectively, and

$$\lambda = \begin{bmatrix} \hat{x}_1 \\ \hat{v} \end{bmatrix}$$

where

$$\hat{v} = M_2 \int (q - A_{21}\hat{x}_1 - D_2\hat{v})\, dt$$

Equation (17) requires the derivative of the output. To eliminate the need for differentiating the output define

$$w = \lambda - My$$

Using this in Eq. (17) results in the estimator dynamics as given in Eq. (3). The estimates of both the state and the disturbance can then be easily obtained from the following:

$$\lambda = w + My \tag{18}$$

$$\hat{x}_2 = y - E\hat{v} \tag{19}$$

Clearly, poles of the estimator are the eigenvalues of the matrix F in Eq. (4). Following a similar procedure to the design of the standard Leunberger observer (see Ref. 13, for example), it can be easily proved that the necessary and sufficient condition for arbitrary assignment of the estimator's poles is that the pair given in Eq. (9) must be completely observable.

To prove asymptotic stability and convergence of the proposed estimator, let us define the following error vectors:

$$e_x = x_1 - \hat{x}_1$$

$$e_v = v - \hat{v}$$

$$e = \begin{pmatrix} e_x \\ e_v \end{pmatrix}$$

Differentiating the expressions for e_x and e_v and using the error definitions already given along with the previous equations, we get the following error dynamics:

$$\dot{e} = \begin{bmatrix} A_{11} - M_1 A_{21} & (D_1 - M_1 D_2) + (M_1 A_{22} - A_{12})E \\ -M_2 A_{21} & M_2(A_{22}E - D_2) \end{bmatrix} e = Fe \tag{20}$$

Since the eigenvalues of F can be assigned arbitrarily, asymptotic stability of the estimator as well as proper rate of convergence can be achieved by the designer.

It is possible to elaborate more on the required observability condition of the pair in Eq. (9), by considering the following.

Theorem 2: The necessary and sufficient condition for observability of the pair $\{A_{21}, A_{11}\}$ is that the pair $\{C, A\}$ has to be observable.

Proof: See Gopinath.[14]

Theorem 3: If the pair in Eq. (9) is completely observable then the pair $\{A_{21}, A_{11}\}$ has to be observable. Specifically, the state as well as the disturbance in the linear system described by Eqs. (1) and (2) can be estimated by using a reduced order PIO if the following necessary conditions are satisfied:

1) The pair $\{C, A\}$ be completely observable.
2) Equation (21) is satisfied.

$$\rho\begin{pmatrix} A & D \\ C & E \end{pmatrix} = n + m \tag{21}$$

where $\rho(\cdot)$ denotes rank of the matrix argument.

Proof: The observability of the pair in Eq. (9) implies that

$$\rho\begin{bmatrix} A_{21} & (D_2 - A_{22}E) \\ A_{21}A_{11} & A_{21}(D_1 - A_{12}E) \\ A_{21}A_{11}^2 & A_{21}A_{11}(D_1 - A_{12}E) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ A_{21}A_{11}^{n-p+m-1} & A_{21}A_{11}^{n-p+m-2}(D_1 - A_{12}E) \end{bmatrix} = n - p + m$$

By the Cayley-Hamilton theorem, the preceding expression can be written as

$$\rho\begin{bmatrix} A_{21} & (D_2 - A_{22}E) \\ A_{21}A_{11} & A_{21}(D_1 - A_{12}E) \\ A_{21}A_{11}^2 & A_{21}A_{11}(D_1 - A_{12}E) \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ A_{21}A_{11}^{n-p-1} & A_{21}A_{11}^{n-p-2}(D_1 - A_{12}E) \end{bmatrix} = n - p + m \tag{22}$$

Now, it is clear that for the rank condition to hold, the pair $\{A_{21}, A_{11}\}$ has to be observable. In addition, the observability of this pair implies that of $\{C, A\}$. Consider now writing Eq. (22) as

$$\rho\left(\begin{bmatrix} 0 & I \\ A_{21} & 0 \\ A_{21}A_{11} & 0 \\ A_{21}A_{11}^2 & 0 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ A_{21}A_{11}^{n-p-2} & 0 \end{bmatrix} \begin{bmatrix} A_{11} & (D_1 - A_{12}E) \\ A_{21} & (D_2 - A_{22}E) \end{bmatrix}\right) = n - p + m \tag{23}$$

then, by Sylvester's inequality theorem we must have

$$\rho\begin{bmatrix} A_{11} & (D_1 - A_{12}E) \\ A_{21} & (D_2 - A_{22}E) \end{bmatrix} = n - p + m \tag{24}$$

Equation (21) can be written in partitioned form as

$$\begin{bmatrix} A_{11} & A_{12} & D_1 \\ A_{21} & A_{22} & D_2 \\ 0 & I & E \end{bmatrix}$$

By simple elementary row and column operation on the matrix just given it can easily be verified that if Eq. (24) is of rank $(n - p + m)$, then that implies that Eq. (21) has to be of rank $(n + m)$.

Corollary 1: For a PIO to exist, there has to be at least as many outputs as there are unknown constant disturbances. That is, $p \geq m$.

Proof: This is a direct consequence of Theorem 3.

III. Illustrative Example

Consider the linearized dynamics of the L-1011 aircraft described in Ref. 11. The state variables are the sideslip angle, bank angle, roll rate, and the yaw rate. The inputs are rudder deflection and aileron deflection. For the sake of illustration, it was assumed that the wind gust is the disturbance acting on the system in the following way:

$$\dot{x} = \begin{bmatrix} -0.2100 & 0.0340 & -0.0011 & -0.9900 \\ 0 & 0 & 1.0 & 0 \\ -5.555 & 0 & -1.8900 & 0.3900 \\ 2.4300 & 0 & -0.0340 & -2.9800 \end{bmatrix} x + \begin{bmatrix} 0.0300 & 0 \\ 0 & 0 \\ 0.3600 & -1.6000 \\ -0.9500 & -0.0320 \end{bmatrix} u + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} v$$

$$y = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0.2 \end{bmatrix} v$$

It is of interest to estimate the sideslip angle as well as the magnitude of the disturbance, perhaps for control purposes. A PIO was designed with its poles located at $(-5.0, -5.5)$. The estimator gain is given by

$$M = \begin{bmatrix} 0 & 0.0180 & 2.0124 \\ 0 & 4.2820 & 9.7886 \end{bmatrix}$$

which would result in the dynamics of the PIO given by

$$\dot{w} = \begin{bmatrix} -5 & 0 \\ 0 & -5.5 \end{bmatrix} w + \begin{bmatrix} 1.9353 & 0.0933 \\ 7.7576 & 7.1644 \end{bmatrix} u + \begin{bmatrix} 0.034 & 0.0112 & -5.0621 \\ 0 & -15.125 & -26.3372 \end{bmatrix} y$$

For illustration purposes $x(0) = [0.1 \quad 0 \quad 0 \quad 0]^T$ was selected and the response of the open-loop system and the estimator was simulated for zero rudder and aileron control inputs and

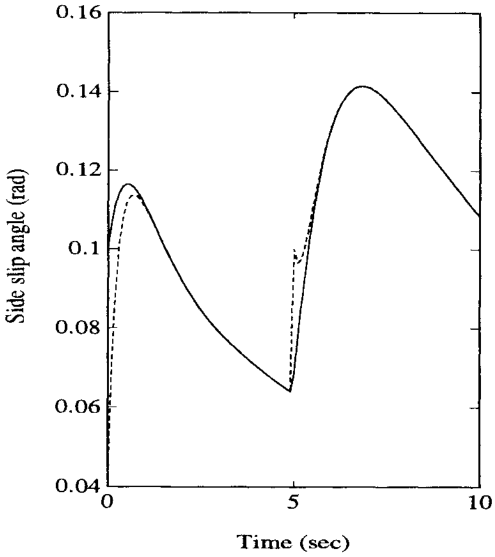

Fig. 1 Actual vs estimated trajectory.

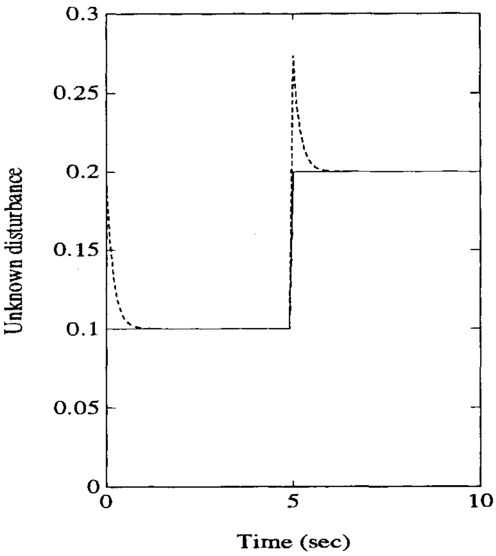

Fig. 2 Actual vs estimated trajectory.

with a stepwise time varying disturbance as shown in Fig. 2. Figures 1 and 2 illustrate the actual state and disturbance trajectories (solid lines) vs their estimates (dotted lines).

IV. Conclusions

A systematic procedure for the design of reduced order robust estimators capable of accurately estimating state as well as constant disturbances acting on the system and the measurement was proposed in this Note. Conditions for the existence of this estimator were outlined.

Acknowledgments

This research was supported in part by the Natural Sciences and Engineering Research Council (NSERC) of Canada, and in part by the Center for Systems Science (CSS).

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Optimal Weighting of a Priori Statistics in Quick-Look Orbit Determination

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Introduction

QUICK-LOOK orbit determination problems are characterized by the need to resolve accurately a vehicle's orbit using a short arc of observations. In such problems, the observational data set is ill-conditioned and statistical information on the nominal solution is commonly incorporated into the solution process to provide stability in the estimation of the vehicle's states. This statistical information is generally the "a priori covariance," which is typically weighted equally with the observational data in conventional minimum-variance (or maximum likelihood) solution methods. Often the a priori information available for certain types of dynamical systems is inaccurate, which can contribute to large errors in the estimates in ill-conditioned problems. Methods developed herein provide for the "optimal" weighting of the a priori covariance

Received May 18, 1992; revision received Oct. 20, 1992; accepted for publication Nov. 6, 1992.

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